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A REFINED NONLINEAR THEORY OF PLATES WITH TRANSVERSE SHEAR DEFORMATION

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Abstract—A higher-order shear deformation theory of plates accounting for the von Karman strains is presented. The theory contains the same dependent unknowns as in the Hencky-Mindlin type first-order shear deformation theory and accounts for parabolic distribution of the transverse shear strains through the thickness of the plate. Exact solutions of simply supported plates are obtained using the linear theory and the results are compared with the exact solutions of 3-D elasticity theory, the first order shear deformation theory, and the classical plate theory. The present theory predicts the deflections, stresses, and frequencies more accurately when compared to the first-order theory and the classical plate theory.

INTRODUCTION

First general solutions to the equations of linear elasticity corresponding to thin plates were presented by Cauchy[1] and Poisson[2] using the methods of series expansion, and by Kirchhoff[3] using certain hypothesis. An expansion in powers of the thickness of the plate was used by Goodier[4] to obtain a general solution in terms of a series of biharmonic functions for a plate subjected to edge tractions. It is well known from experimental observations that the Poisson–Kirchhoff theory of plates, in which it is assumed that normals to the midplane before deformation remain straight and normal to the plane after deformation, underpredicts deflections and overpredicts natural frequencies. These results are due to the neglect of transverse shear strains in the classical plate theory (CPT).

Refined plate theories, due to Levy[5], Reissner[6, 7], Hencky[8], Mindlin[9], and Kromm[10] are improvements of the classical plate theory in that they include the effect of transverse shear deformation (see[11]). In the Hencky–Mindlin theories the displacements are expanded in powers of the thickness of the plate (see[12–14]). Extensions of the Kirchhoff–von Karman theory[15], a geometrically nonlinear theory associated with the classical plate theory, to refined plate theories were considered by Reissner[16, 17] and Medwadowski[18]. Extension of the Kromm's theory to geometrically nonlinear analysis, in the sense of von Karman, is due to Schmidt[19].

These higher-order theories are cumbersome and computationally more demanding, because, with each additional power of the thickness coordinate, an additional dependent unknown is introduced into the theory. Further, these theories require an arbitrary correction to the transverse shear stiffnesses, and the transverse shear stresses do not satisfy the conditions of zero transverse shear stresses on the top and bottom surfaces of the plate. Of course, the Reissner–Kromm theories satisfy the stress free conditions, but these are based on the stress fields. Thus, need exists for the development of a higher-order shear deformation theory that avoids the shear correction factors, and accurately predicts transverse shear stresses. Levinson[20] considered such a plate theory, in which the in-plane displacements are expanded as cubic functions of the thickness coordinate. Unfortunately, both Levinson[20] and Schmidt[19] used variationally inconsistent set of equilibrium equations (they used the equilibrium equations of the classical plate theory), and therefore did not correctly account for all of the strain energy associated with the displacement field.

The present theory accounts for the cubic variation of the in-plane displacements through the plate thickness, the von Karman strains, and transverse shear strains which vanish on the top and bottom faces of the plate. The equations of motion are derived using Hamilton's principle, and therefore they are consistent with the assumed displacement field. In order to illustrate the accuracy of the present theory, the exact solutions of the linear theory are presented for bending and vibration of simply supported, homogeneous, isotropic and orthotropic rectangular plates. Comparison of the present solutions with the 3D elasticity solutions shows that the present theory yields more accurate stresses and natural frequencies than the first-order shear deformation theory.

KINEMATICS

We begin with the displacement field in which the displacements along the x- and y-directions are expanded as cubic functions of the thickness coordinate, and the transverse deflection is assumed to be constant through thickness:

$$\begin{aligned} u_1(x,y,z,t) &= u(x,y,t) + z\psi_x(x,y,t) + z^2\xi_x(x,y,t) + z^3\zeta_x(x,y,t) \\ u_2(x,y,z,t) &= v(x,y,t) + z\psi_y(x,y,t) + z^2\xi_y(x,y,t) + z^3\zeta_y(x,y,t) \\ u_3(x,y,t) &= w(x,y,t). \end{aligned} \tag{1}$$

Here u, v, and w denote the displacements of a point (x, y) on the midplane, and ψ_x and ψ_y are the rotations of normals to midplane about the y and x axes, respectively. The functions ξ_x, ζ_x, ξ_y, and ζ_y will be determined using the condition that the transverse shear stresses, $\sigma_{xz} = \sigma_5$ and $\sigma_{yz} = \sigma_4$ vanish on the plate top and bottom surfaces:

$$\sigma_5\left(x, y, \pm\frac{h}{2}, t\right) = 0, \quad \sigma_4\left(x, y, \pm\frac{h}{2}, t\right) = 0 \tag{2}$$

these conditions are equivalent to the requirement that the corresponding strains be zero on these surfaces. We have

$$\begin{aligned} \epsilon_5 &= \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \psi_x + 2z\xi_x + 3z^2\zeta_x + \frac{\partial w}{\partial x} \\ \epsilon_4 &= \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \psi_y + 2z\xi_y + 3z^2\zeta_y + \frac{\partial w}{\partial y}. \end{aligned} \tag{3}$$

Setting $\epsilon_5(x, y, \pm h/2, t)$ and $\epsilon_4(x, y, \pm h/2, t)$ to zero, we obtain

$$\begin{aligned} &\xi_x = 0, \quad \xi_y = 0 \\ &\zeta_x = -\frac{4}{3h^2}\left(\frac{\partial w}{\partial x} + \psi_x\right), \quad \zeta_y = -\frac{4}{3h^2}\left(\frac{\partial w}{\partial y} + \psi_y\right). \end{aligned} \tag{4}$$

The displacement field in eqn (1) becomes

$$\begin{aligned} u_1 &= u + z\left[\psi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_x + \frac{\partial w}{\partial x}\right)\right] \\ u_2 &= v + z\left[\psi_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_y + \frac{\partial w}{\partial y}\right)\right] \\ u_3 &= w. \end{aligned} \tag{5}$$

One should note that, although cubic variation of the in-plane displacements through thickness is accounted, the displacement field in eqn (5) contains the same number of dependent variables as in the first-order shear deformation theory. This is an attractive feature from finite-element modeling considerations.

The von Karman strains associated with the displacement field in eqn (5) are

$$\begin{aligned}
\epsilon_1 &= \epsilon_1{}^0 + z(\kappa_1{}^0 + z^2\kappa_1{}^2)\\
\epsilon_2 &= \epsilon_2{}^0 + z(\kappa_2{}^0 + z^2\kappa_2{}^2)\\
\epsilon_3 &= 0\\
\epsilon_4 &= \epsilon_4{}^0 + z^2\kappa_4{}^2\\
\epsilon_5 &= \epsilon_5{}^0 + z^2\kappa_5{}^2\\
\epsilon_6 &= \epsilon_6{}^0 + z(\kappa_6{}^0 + z^2\kappa_6{}^2)
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
&\epsilon_1{}^0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\ \kappa_1{}^0 = \frac{\partial \psi_x}{\partial x},\ \kappa_1{}^2 = -\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\\
&\epsilon_2{}^0 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\ \kappa_2{}^0 = \frac{\partial \psi_y}{\partial y},\ \kappa_2{}^2 = -\frac{4}{3h^2}\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\\
&\epsilon_4{}^0 = \psi_y + \frac{\partial w}{\partial y},\ \kappa_4{}^2 = -\frac{4}{h^2}\left(\psi_y + \frac{\partial w}{\partial y}\right)\\
&\epsilon_5{}^0 = \psi_x + \frac{\partial w}{\partial x},\ \kappa_5{}^2 = -\frac{4}{h^2}\left(\psi_x + \frac{\partial w}{\partial x}\right)\\
&\epsilon_6{}^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},\ \kappa_6{}^0 = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x},\ \kappa_6{}^2 = -\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right).
\end{aligned} \tag{7}$$

It is interesting to note that the new strain components contain higher-order derivatives of the transverse deflection.

CONSTITUTIVE EQUATIONS

For a plate of constant thickness h and made of an orthotropic material (i.e. the plate possesses a plane of elastic symmetry parallel to the x–y plane) the constitutive equations can be written as

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_6\end{Bmatrix},\ \begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix} = \begin{bmatrix} Q_{44} & 0\\ 0 & Q_{55}\end{bmatrix}\begin{Bmatrix}\epsilon_4\\ \epsilon_5\end{Bmatrix} \tag{8a}$$

where Q_{ij} are the plane-stress reduced elastic constants in the material axes of the plate:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}},\ Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}},\ Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \tag{8b}$$

$$Q_{44} = G_{23},\ Q_{55} = G_{13},\ Q_{66} = G_{12}.$$

EQUATIONS OF MOTION

Here we use Haminton's principle to derive the equations of motion appropriate for the displacement field (5) and constitutive equations (8). The principle can be stated in analytical form as (see Reddy and Rasmussen[21])

$$
\begin{aligned}
0 = & -\int_0^t \left[\int_{-h/2}^{h/2}\int_R (\sigma_1\delta\epsilon_1 + \sigma_2\delta\epsilon_2 + \sigma_6\delta\epsilon_6 + \sigma_4\delta\epsilon_4 + \sigma_5\delta\epsilon_5)\,\mathrm{d}A\,\mathrm{d}z + \int_R q\delta w\,\mathrm{d}x\,\mathrm{d}y\right]\mathrm{d}t \\
& + \frac{\delta}{2}\int_0^t\int_{-h/2}^{h/2}\int_R \rho[(\dot{u}_1)^2 + (\dot{u}_2)^2 + (\dot{u}_3)^2]\,\mathrm{d}A\,\mathrm{d}z\,\mathrm{d}t \\
= & -\int_0^t\left[\int_R\left\{N_1\left(\frac{\partial\delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}\right) + M_1\frac{\partial\delta\psi_x}{\partial x} + P_1\left[-\frac{4}{3h^2}\left(\frac{\partial\delta\psi_x}{\partial x} + \frac{\partial^2\delta w}{\partial x^2}\right)\right]\right.\right. \\
& + N_2\left(\frac{\partial\delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial y}\right) + M_2\frac{\partial\delta\psi_y}{\partial y} + P_2\left[-\frac{4}{3h^2}\left(\frac{\partial\delta\psi_y}{\partial y} + \frac{\partial^2\delta w}{\partial y^2}\right)\right] \\
& + N_6\left(\frac{\partial\delta u}{\partial y} + \frac{\partial\delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial y} + \frac{\partial\delta w}{\partial x}\frac{\partial w}{\partial y}\right) + M_6\left(\frac{\partial\delta\psi_x}{\partial y} + \frac{\partial\delta\psi_y}{\partial x}\right) \\
& + P_6\left[-\frac{4}{3h^2}\left(\frac{\partial\delta\psi_x}{\partial y} + \frac{\partial\delta\psi_y}{\partial x} + 2\frac{\partial^2\delta w}{\partial x\partial y}\right)\right] + Q_2\left(\delta\psi_y + \frac{\partial\delta w}{\partial y}\right) \\
& + R_2\left[-\frac{4}{h^2}\left(\delta\psi_y + \frac{\partial\delta w}{\partial y}\right)\right] + Q_1\left(\delta\psi_x + \frac{\partial\delta w}{\partial x}\right) + R_1\left[-\frac{4}{h^2}\left(\delta\psi_x + \frac{\partial\delta w}{\partial x}\right)\right] \\
& \left. + q\delta w\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int_0^t\int_R\left\{\delta u\left[I_1\ddot{u} + \left(I_2 - \frac{4}{3h^2}I_4\right)\ddot{\psi}_x + \left(-\frac{4}{3h^2}\right)I_4\frac{\partial\ddot{w}}{\partial x}\right]\right. \\
& + \delta v\left[I_1\ddot{v} + \left(I_2 - \frac{4}{3h^2}I_4\right)\ddot{\psi}_y - \frac{4}{3h^2}I_4\frac{\partial\ddot{w}}{\partial y}\right] + \delta w\left[I_1\ddot{w} - \left(\frac{4}{3h^2}\right)^2 I_7\left(\frac{\partial^2\ddot{w}}{\partial x^2} + \frac{\partial^2\ddot{w}}{\partial y^2}\right)\right. \\
& \left. + \frac{4}{3h^2}I_4\left(\frac{\partial\ddot{u}}{\partial x} + \frac{\partial\ddot{v}}{\partial y}\right) + \frac{4}{3h^2}\left(I_5 - \frac{4}{3h^2}I_7\right)\left(\frac{\partial\ddot{\psi}_x}{\partial x} + \frac{\partial\ddot{\psi}_y}{\partial y}\right)\right] \\
& + \delta\psi_x\left[\left(I_2 - \frac{4}{3h^2}I_4\right)\ddot{u} + \left(I_3 - \frac{8}{3h^2}I_5 + \frac{16}{3h^4}I_7\right)\ddot{\psi}_x\right. \\
& \left. - \frac{4}{3h^2}\left(I_5 - \frac{4}{3h^2}I_7\right)\frac{\partial\ddot{w}}{\partial x}\right] + \delta\psi_y\left[\left(I_2 - \frac{4}{3h^2}I_4\right)\ddot{v}\right. \\
& \left.\left.\left. + \left(I_3 - \frac{8}{3h^2}I_5 + \frac{16}{9h^4}I_7\right)\ddot{\psi}_y - \frac{4}{3h^2}\left(I_5 - \frac{4}{3h^2}I_7\right)\frac{\partial\ddot{w}}{\partial y}\right]\right\}\mathrm{d}A\right]\mathrm{d}t
\end{aligned}
\tag{9}
$$

where the stress resultants N_i, M_i, P_i, Q_i and R_i are defined by

$$
\begin{aligned}
(N_i, M_i, P_i) &= \int_{-h/2}^{h/2}\sigma_i(1, z, z^3)\,\mathrm{d}z\,(i = 1, 2, 6) \\
(Q_2, R_2) &= \int_{-h/2}^{h/2}\sigma_4(1, z^2)\,\mathrm{d}z \\
(Q_1, R_1) &= \int_{-h/2}^{h/2}\sigma_5(1, z^2)\,\mathrm{d}z
\end{aligned}
\tag{10}
$$

and the inertias $I_i (i = 1, 2, 3, 4, 5, 7)$ are defined by

$$
(I_1, I_2, I_3, I_4, I_5, I_7) = \int_{-h/2}^{h/2}\rho(1, z, z^2, z^3, z^4, z^6)\,\mathrm{d}z. \tag{11}
$$

Integrating the expressions in eqn (9) by parts, and collecting the coefficients of δu, δv, δw, $\delta\psi_x$, and $\delta\psi_y$, we obtain the following equations of motion:

$$\delta u\colon \frac{\partial N_1}{\partial x}+\frac{\partial N_6}{\partial y}=I_1\ddot{u}+\bar{I}_2\ddot{\psi}_x-\frac{4}{3h^2}I_4\frac{\partial \ddot{w}}{\partial x}$$

$$\delta v\colon \frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}=I_1\ddot{v}+\bar{I}_2\ddot{\psi}_y-\frac{4}{3h^2}I_4\frac{\partial \ddot{w}}{\partial y}$$

$$\begin{aligned}\partial w\colon & \frac{\partial Q_1}{\partial x}+\frac{\partial Q_2}{\partial y}+\frac{\partial}{\partial x}\left(N_1\frac{\partial w}{\partial x}+N_6\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_6\frac{\partial w}{\partial x}+N_2\frac{\partial w}{\partial y}\right)\\ &+q-\frac{4}{h^2}\left(\frac{\partial R_1}{\partial x}+\frac{\partial R_2}{\partial y}\right)+\frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial x^2}+2\frac{\partial^2 P_6}{\partial x\partial y}+\frac{\partial^2 P_2}{\partial y^2}\right)\\ &=I_1\ddot{w}-\left(\frac{4}{3h^2}\right)^2 I_7\left(\frac{\partial^2\ddot{w}}{\partial x^2}+\frac{\partial^2\ddot{w}}{\partial y^2}\right)+\frac{4}{3h^2}I_4\left(\frac{\partial\ddot{u}}{\partial x}+\frac{\partial\ddot{v}}{\partial y}\right)+\frac{4}{3h^2}\bar{I}_5\left(\frac{\partial\ddot{\psi}_x}{\partial x}+\frac{\partial\ddot{\psi}_y}{\partial y}\right)\end{aligned}$$

$$\delta\psi_x\colon \frac{\partial M_1}{\partial x}+\frac{\partial M_6}{\partial y}-Q_1+\frac{4}{h^2}R_1-\frac{4}{3h^2}\left(\frac{\partial P_1}{\partial x}+\frac{\partial P_6}{\partial y}\right)=\bar{I}_2\ddot{u}+\bar{I}_3\ddot{\psi}_x-\frac{4}{3h^2}\bar{I}_5\frac{\partial\ddot{w}}{\partial x}$$

$$\delta\psi_y\colon \frac{\partial M_6}{\partial x}+\frac{\partial M_2}{\partial y}-Q_2+\frac{4}{h^2}R_2-\frac{4}{3h^2}\left(\frac{\partial P_6}{\partial x}+\frac{\partial P_2}{\partial y}\right)=\bar{I}_2\ddot{v}+\bar{I}_3\ddot{\psi}_y-\frac{4}{3h^2}\bar{I}_5\frac{\partial\ddot{w}}{\partial y} \tag{12}$$

where

$$\bar{I}_2=I_2-\frac{4}{3h^2}I_4,\quad \bar{I}_5=I_5-\frac{4}{3h^2}I_7,$$

$$\bar{I}_3=I_3-\frac{8}{3h^2}I_5+\frac{16}{9h^4}I_7. \tag{13}$$

The boundary conditions are of the form: specify

$$\left.\begin{array}{r} u_n \text{ or } N_n\\ u_{ns} \text{ or } N_{ns}\\ w \text{ or } Q_n\\ \dfrac{\partial w}{\partial n} \text{ or } P_n\\ \psi_n \text{ or } M_n\\ \psi_{ns} \text{ or } M_{ns}\end{array}\right\}\quad \text{on } \Gamma \tag{14}$$

where Γ is the boundary of the midplane Ω of the plate, and

$$\begin{aligned} u_n&=un_x+vn_y,\; u_{ns}=-un_y+vn_x\\ N_n&=N_1n_x^2+N_2n_y^2+2N_6n_xn_y\\ N_{ns}&=(N_2-N_1)n_xn_y+N_6(n_x^2-n_y^2)\\ M_n&=\hat{M}_1n_x^2+\hat{M}_2n_y^2+2\hat{M}_6n_xn_y\\ M_{ns}&=(\hat{M}_2-\hat{M}_1)n_xn_y+\hat{M}_6(n_x^2-n_y^2)\\ Q_n&=\hat{Q}_1n_x+\hat{Q}_2n_y-\frac{4}{3h^2}\frac{\partial P_{ns}}{\partial s}+\bar{N}_n\end{aligned} \tag{15}$$

$$\hat{M}_i = M_i - \frac{4}{3h^2} P_i \, (i = 1, 2, 6)$$

$$\hat{Q}_i = Q_i - \frac{4}{h^2} R_i \, (i = 1, 2)$$

$$\bar{N}_n = \left(N_1 \frac{\partial w}{\partial x} + N_6 \frac{\partial w}{\partial y}\right) n_x + \left(N_6 \frac{\partial w}{\partial x} + N_2 \frac{\partial w}{\partial y}\right) n_y$$

$$\frac{\partial}{\partial n} = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial s} = n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x}$$

and P_n and P_{ns} are defined by expressions analogous to N_n and N_{ns}, respectively. This completes the derivation of the governing equations. An examination of the boundary conditions in eqn (14) shows that both ψ_n and $\partial w/\partial n$ are geometric boundary conditions in the present theory. Consequently, one should use interpolation functions that guarantee interelement continuity of slopes in the finite-element modeling of the theory.

The resultants defined in eqn (10) can be related to the total strains in eqn (6) by the following equations:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ & A_{22} & 0 \\ \text{sym.} & & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1^0 \\ \epsilon_2^0 \\ \epsilon_6^0 \end{Bmatrix}$$

$$\begin{Bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} \\ \\ \begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ \text{sym.} & & D_{66} \end{bmatrix} & \begin{bmatrix} F_{11} & F_{12} & 0 \\ & F_{22} & 0 \\ \text{sym.} & & F_{66} \end{bmatrix} \\ \\ \text{sym.} & \begin{bmatrix} H_{11} & H_{12} & 0 \\ & H_{22} & 0 \\ \text{sym.} & & H_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{Bmatrix} \\ \\ \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_4^0 \\ \epsilon_5^0 \end{Bmatrix}, \begin{Bmatrix} R_2 \\ R_1 \end{Bmatrix} = \begin{bmatrix} F_{44} & 0 \\ 0 & F_{55} \end{bmatrix} \begin{Bmatrix} \kappa_4^2 \\ \kappa_5^2 \end{Bmatrix} \tag{16}$$

where A_{ij}, B_{ij}, etc. are the plate stiffnesses, defined by

$$(A_{ij}, D_{ij}, F_{ij}, H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z^2, z^4, z^6)\, dz \, (i, j = 1, 2, 6)$$

$$(A_{ij}, D_{ij}, F_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z^2, z^4)\, dz \, (i, j = 4, 5) \tag{17}$$

or

$$\begin{aligned} &A_{ij} = Q_{ij}, \; D_{ij} = Q_{ij}(h^3/12) \\ &F_{ij} = Q_{ij}(h^5/80), \; H_{ij} = Q_{ij}(h^7/448) \\ &A_{44} = G_{23}h, \; A_{55} = G_{13}h \\ &D_{44} = G_{23}(h^3/12), \; D_{55} = G_{13}(h^3/12) \\ &F_{44} = G_{23}(h^5/80), \; F_{55} = G_{13}(h^5/80). \end{aligned} \tag{18}$$

EXACT SOLUTIONS FOR SIMPLY SUPPORTED RECTANGULAR PLATES

The exact analytical solution of the nonlinear partial differential equations in eqn (12) is an impossible task. Even the linear equations do not allow an exact solution for all geometries and boundary conditions. Here we consider the exact solutions of eqns (12) and (13) for infinitesimal displacement theory of simply supported, rectangular plates. Since the coupling between stretching and bending is zero for the linear theory, we consider only the flexural displacements and natural frequencies. The following "simply-supported" boundary conditions are assumed (a and b are the plane-form dimensions and the origin of the coordinate system is taken at the lower left corner of the plate):

$$\begin{aligned} &w(x,0)=w(x,b)=w(0,y)=w(a,y)=0\\ &P_2(x,0)=P_2(x,b)=P_1(0,y)=P_1(a,y)=0\\ &M_2(x,0)=M_2(x,b)=M_1(0,y)=M_1(a,y)=0\\ &\psi_x(x,0)=\psi_x(x,b)=\psi_y(0,y)=\psi_y(a,y)=0. \end{aligned} \tag{19}$$

The resultants of eqn (16) can be expressed in terms of the generalized displacements, for the case of infinitesimal displacements, as

$$\begin{aligned}
N_1&=A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}\\
N_2&=A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}\\
N_6&=A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\
M_1&=D_{11}\frac{\partial\psi_x}{\partial x}+D_{12}\frac{\partial\psi_y}{\partial y}+F_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\\
M_2&=D_{12}\frac{\partial\psi_x}{\partial x}+D_{22}\frac{\partial\psi_y}{\partial y}+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\\
M_6&=D_{66}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)\\
Q_2&=A_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)+D_{44}\left(-\frac{4}{h^2}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\\
Q_1&=A_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)+D_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x+\frac{\partial w}{\partial k}\right)\\
P_1&=F_{11}\frac{\partial\psi_x}{\partial x}+F_{12}\frac{\partial\psi_y}{\partial y}+H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\\
P_2&=F_{12}\frac{\partial\psi_x}{\partial x}+F_{22}\frac{\partial\psi_y}{\partial y}+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\\
P_6&=F_{66}\left(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\right)+H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)\\
R_2&=D_{44}\left(\frac{\partial w}{\partial y}+\psi_y\right)+F_{44}\left(-\frac{4}{h^2}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\\
R_1&=D_{55}\left(\frac{\partial w}{\partial x}+\psi_x\right)+F_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x+\frac{\partial w}{\partial x}\right).
\end{aligned} \tag{20}$$

The last three equations in eqn (12), for the linear theory, can be expressed in terms of the displacements as

$$\frac{4}{3h^2}\left[F_{11}\frac{\partial^3\psi_x}{\partial x^3}+H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_x}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}\right)+F_{12}\frac{\partial^3\psi_y}{\partial x^2\partial y}\right.$$
$$+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_y}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)+F_{12}\frac{\partial^3\psi_x}{\partial y^2\partial x}$$
$$+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_x}{\partial y^2\partial x}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)+F_{22}\frac{\partial^3\psi_y}{\partial y^3}$$
$$+H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_y}{\partial y^3}+\frac{\partial^4 w}{\partial y^4}\right)+2F_{66}\left(\frac{\partial^3\psi_y}{\partial x^2\partial y}+\frac{\partial^3\psi_x}{\partial y^2\partial x}\right)$$
$$\left.+2H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_x}{\partial y^2\partial x}+\frac{\partial^3\psi_y}{\partial x^2\partial y}+2\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\right]$$
$$-\frac{4}{h^2}\left[D_{55}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\psi_x}{\partial x}\right)+F_{55}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right.$$
$$\left.+D_{44}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\psi_y}{\partial y}\right)+F_{44}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right]$$
$$+\left[A_{55}\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+D_{55}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right.$$
$$\left.+A_{44}\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+D_{44}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right]+q$$
$$=I_1\frac{\partial^2 w}{\partial t^2}-\left(\frac{4}{3h^2}\right)^2 I_7\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+\frac{4}{3h^2}I_5\left(\frac{\partial^3\psi_x}{\partial x\partial t^2}+\frac{\partial^3\psi_y}{\partial y\partial t^2}\right) \tag{21a}$$

$$D_{11}\frac{\partial^2\psi_x}{\partial x^2}+D_{12}\frac{\partial^2\psi_y}{\partial x\partial y}+F_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)$$
$$+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+D_{66}\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}\right)$$
$$+F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)-\left[A_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)\right.$$
$$\left.+D_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x+\frac{\partial w}{\partial x}\right)\right]-\frac{4}{3h^2}\left[F_{11}\frac{\partial^2\psi_x}{\partial x^2}+H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x^2}\right.\right.$$
$$\left.+\frac{\partial^3 w}{\partial x^3}\right)+F_{12}\frac{\partial^2\psi_y}{\partial x\partial y}+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)$$
$$\left.+F_{66}\left(\frac{\partial^2\psi_y}{\partial x\partial y}+\frac{\partial^2\psi_x}{\partial y^2}\right)+H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)\right]$$
$$+\frac{4}{h^2}\left[D_{55}\left(\frac{\partial w}{\partial x}+\psi_x\right)+F_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x+\frac{\partial w}{\partial x}\right)\right]$$
$$=I_3\frac{\partial^2\psi_x}{\partial t^2}-\frac{4}{3h^2}I_5\frac{\partial^3 w}{\partial x\partial t^2} \tag{21b}$$

$$D_{66}\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^2\psi_y}{\partial x^2}\right)+F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^2\psi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)$$

$$+D_{12}\frac{\partial^2\psi_x}{\partial x\partial y}+D_{22}\frac{\partial^2\psi_y}{\partial y^2}+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)$$

$$+F_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)-\left[A_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)+D_{44}\left(-\frac{4}{h^2}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\right]$$

$$-\frac{4}{3h^2}\left[F_{66}\left(\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_x}{\partial y\partial x}\right)+H_{66}\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^2\psi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)\left(-\frac{4}{3h^2}\right)\right.$$

$$+F_{12}\frac{\partial^2\psi_x}{\partial x\partial y}+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)$$

$$\left.+F_{22}\frac{\partial^2\psi_y}{\partial y^2}+H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)\right]$$

$$+\frac{4}{h^2}\left[D_{44}\left(\frac{\partial w}{\partial y}+\psi_y\right)+F_{44}\left(-\frac{4}{h^2}\right)\left(\frac{\partial w}{\partial y}+\psi_y\right)\right]$$

$$=\bar{I}_3\frac{\partial^2\psi_y}{\partial t^2}-\frac{4}{3h^2}\bar{I}_5\frac{\partial^3 w}{\partial y\partial t^2}. \tag{21c}$$

Following the Navier solution procedure, we assume the following solution form that satisfies the boundary conditions in eqn (19),

$$w=\sum_{m,n=1}^{\infty}W_{mn}\sin\alpha x\sin\beta y\, e^{-i\omega t}$$

$$\psi_x=\sum_{m,n=1}^{\infty}X_{mn}\cos\alpha x\sin\beta y\, e^{-i\omega t} \tag{22}$$

$$\psi_y=\sum_{m,n=1}^{\infty}Y_{mn}\sin\alpha x\cos\beta y\, e^{-i\omega t}$$

where $\alpha=m\pi/a$ and $\beta=n\pi/b$, and ω is the frequency of the natural vibration. Further we assume that the applied transverse load, q, can be expanded in the double-Fourier series as

$$q=\sum_{m,n=1}^{\infty}Q_{mn}\sin\alpha x\sin\beta y. \tag{23}$$

Substituting eqns (22) and (23) into eqn (21), and collecting the coefficients, we obtain

$$\begin{bmatrix}M_{11}&M_{12}&M_{13}\\M_{12}&M_{22}&M_{23}\\M_{13}&M_{23}&M_{33}\end{bmatrix}\begin{Bmatrix}\ddot{W}_{mn}\\\ddot{X}_{mn}\\\ddot{Y}_{mn}\end{Bmatrix}+\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{12}&c_{22}&c_{23}\\c_{13}&c_{23}&c_{33}\end{bmatrix}\begin{Bmatrix}W_{mn}\\X_{mn}\\Y_{mn}\end{Bmatrix}=\begin{Bmatrix}Q_{mn}\\0\\0\end{Bmatrix} \tag{24}$$

for any fixed values of m and n. The elements $c_{ij}=c_{ji}$ and $M_{ij}=M_{ji}$ of the coefficient matrices $[c]$ and $[M]$ are given by

$$c_{11} = \alpha^2 A_{55} + \beta^2 A_{44} - \frac{8}{h^2}(\alpha^2 D_{55} + \beta^2 D_{44})$$

$$+ \left(\frac{4}{h^2}\right)^2 (\alpha^2 F_{55} + \beta^2 F_{44}) + \left(\frac{4}{3h^2}\right)^2 [\alpha^4 H_{11} + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + \beta^4 H_{22}]$$

$$c_{12} = \alpha A_{55} - \frac{8}{h^2}\alpha D_{55} + \left(\frac{4}{h^2}\right)^2 \alpha F_{55} - \frac{4}{3h^2}[\alpha^3 F_{11} + \alpha\beta^2(F_{12} + 2F_{66})]$$

$$+ \left(\frac{4}{3h^2}\right)^2 [\alpha^3 H_{11} + \alpha\beta^2(H_{12} + 2H_{66})]$$

$$c_{13} = \beta A_{44} - \frac{8}{h^2}\beta D_{44} + \left(\frac{4}{h^2}\right)^2 \beta F_{44} - \frac{4}{3h^2}[\alpha^2\beta(F_{12} + 2F_{66}) + \beta^3 F_{22}]$$

$$+ \left(\frac{4}{3h^2}\right)^2 [\alpha^2\beta(H_{12} + 2H_{66}) + \beta^3 H_{22}]$$

$$c_{22} = A_{55} + \alpha^2 D_{11} + \beta^2 D_{66} - \frac{8}{h^2} D_{55} + \left(\frac{4}{h^2}\right)^2 F_{55}$$

$$- \frac{8}{3h^2}(\alpha^2 F_{11} + \beta^2 F_{66}) + \left(\frac{4}{3h^2}\right)^2 (\alpha^2 H_{11} + \beta^2 H_{66})$$

$$c_{23} = \alpha\beta\left[D_{12} + D_{66} - \frac{8}{3h^2}(F_{12} + F_{66}) + \left(\frac{4}{3h^2}\right)^2 (H_{12} + H_{66})\right]$$

$$c_{33} = A_{44} + \alpha^2 D_{66} + \beta^2 D_{22} - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44}$$

$$- \frac{8}{3h^2}(\alpha^2 F_{66} + \beta^2 F_{22}) + \left(\frac{4}{3h^2}\right)^2 (\beta^2 H_{22} + \alpha^2 H_{66})$$

$$M_{11} = I_1 + I_7\left(\frac{4}{3h^2}\right)(\alpha^2 + \beta^2)$$

$$M_{12} = -\frac{4}{3h^2}\bar{I}_5\alpha, \quad M_{13} = -\frac{4}{3h^2}\bar{I}_5\beta$$

$$M_{22} = \bar{I}_3, \quad M_{33} = \bar{I}_3, \quad M_{23} = 0. \tag{25}$$

For static bending eqn (24) takes the form

$$[C]\begin{Bmatrix} W \\ X \\ Y \end{Bmatrix} = \begin{Bmatrix} Q \\ 0 \\ 0 \end{Bmatrix} \tag{26}$$

and for free vibration we have

$$[c]\begin{Bmatrix} W \\ X \\ Y \end{Bmatrix} = \omega^2 [M]\begin{Bmatrix} W \\ X \\ Y \end{Bmatrix} \tag{27}$$

where

$$c_{11} = \frac{8}{15} h(\alpha^2 G_{13} + \beta^2 G_{23}) + \frac{h^3}{12}[\alpha^4 Q_{11} + 2(2G_{12} + Q_{12})\alpha^2\beta^2 + \beta^4 Q_{22}]$$

$$c_{12} = \frac{8}{15} h\alpha G_{12} - \frac{4h^3}{315}[\alpha^3 Q_{11} + \alpha\beta^2(2G_{12} + Q_{12})]$$

$$c_{13} = \frac{8}{15} h\beta G_{23} - \frac{4h^3}{315}[\alpha^2\beta(2G_{12} + Q_{12}) + \beta^3 Q_{22}]$$

$$c_{22} = \frac{8}{15} hG_{13} + \frac{17h^3}{315}(\alpha^2 Q_{11} + \beta^2 G_{12})$$

$$c_{23} = \frac{17\alpha\beta h^3}{315}(G_{12} + Q_{12})$$

$$c_{33} = \frac{8}{15} hG_{23} + \frac{17h^3}{315}(\alpha^2 G_{12} + \beta^2 Q_{22})$$

and Q_{ij} are given by eqn (8b).

The static solution is given by eqn (22) with $t = 0$ and (W, X, Y) from eqn (26). Note that for uniformly distributed load Q_{mn} is given by

$$Q_{mn} = \begin{cases} \dfrac{16q_0}{mn\pi^2}, m, n = 1, 3, \ldots \\ 0, m, n = 2, 4, \ldots \end{cases} \tag{28}$$

NUMERICAL RESULTS

Bending

Numerical results are presented in Tables 1 and 2 for homogeneous isotropic ($\nu = 0.3$) and orthotropic plates under uniformly distributed transverse load of intensity q_0. The

Table 1. Comparison of deflections and stresses in isotropic ($\nu = 0.3$) plates under uniformly distributed transverse load ($m, n = 1, 2, \ldots, 19$)

							Constitutive Eqn.		Equilibrium Eqn.	
b/a	a/h	source	$\bar{w}$	$\pm\bar{\sigma}_1$	$\pm\bar{\sigma}_2$	$\mp\bar{\sigma}_6$	$-\bar{\sigma}_4$	$-\bar{\sigma}_5$	$-\bar{\sigma}_4$	$-\bar{\sigma}_5$
1	5	HSDT	0.0535	0.2944	0.2944	0.2112	0.4840	0.4840	0.3703	0.3703
				(0.2949)⁺	(0.2949)	(0.2124)	(0.4871)	(0.4871)	(0.3324)	(0.3324)
		FSDT	0.0536	0.2873	0.2873	0.1946	0.3928	0.3928	0.4909	0.4909
	10	HSDT	0.0467	0.2890	0.2890	0.1990	0.4890	0.4890	0.4543	0.4543
				(0.2893)	(0.2893)	(0.1996)	(0.4937)	(0.4937)	(0.4417)	(0.4417)
		FSDT	0.0467	0.2873	0.2873	0.1946	0.3928	0.3928	0.4909	0.4909
	100	HSDT	0.0444	0.2873	0.2873	0.1947	0.4909	0.4909	0.4905	0.4905
				(0.2874)	(0.2874)	(0.1948)	(0.4965)	(0.4965)	(0.4959)	(0.4959)
		FSDT	0.444	0.2873	0.2873	0.1946	0.3928	0.3928	0.4909	0.4909
							(0.3972)	(0.3972)	(0.4965)	(0.4965)
	CPT		0.0444	0.2873	0.2873	0.1946	0.0	0.0	0.4909	0.4909
									(0.4965)	(0.4965)
2	5	HSDT	0.1248	0.6202	0.2818	0.2927	0.6745	0.5201	0.5615	0.4569
		FSDT	0.1248	0.6100	0.2779	0.2769	0.5451	0.4192	0.6813	0.6813
	10	HSDT	0.1142	0.6125	0.2789	0.2809	0.6794	0.5230	0.6448	0.5051
		FSDT	0.1142	0.6100	0.2779	0.2769	0.5451	0.4192	0.6813	0.5240
	100	HSDT	0.1106	0.6100	0.2779	0.2769	0.6813	0.5240	0.6809	0.5238
		FSDT	0.1106	0.6100	0.2779	0.2769	0.5451	0.4192	0.6813	0.5240
	CPT		0.1106	0.6100	0.2779	0.2769	0.0	0.0	0.6813	0.5240

⁺numbers in parenthesis were obtained by m,n = 1,3,...,29 in the series of Eq. (22).

Table 2. Comparison of deflections and stresses in orthotropic square plates under uniform transverse load ($m, n = 1, 3, \ldots, 19$)

		$c_{11}w/hq_0$				σ_1/q_0				σ_5/q_0(*)			
b/a	h/a	Exact†	HSDPT	FSDPT	CPT	Exact	HSDPT	FSDPT	CPT	Exact	HSDPT	FSDPT	CPT
2	0.05	21,542	21,542	21,542	21,210	262.67	262.6	262.0	262.2	14.048 (13.57)	13.98 (14.00)	11.20 (14.00)	0.0
	0.10	1,408.5	1,408.5	1,408.5	1,326	65.975	65.95	65.38	65.55	6.927 (6.229)	6.958 (6.998)	5.599 (6.998)	0.0
	0.14	387.23	387.5	387.6	345.1	33.862	33.84	33.27	33.44	4.878 (4.027)	4.944 (4.997)	3.998 (4.999)	0.0
1	0.05	10,443	10450.	10450.	10250.	144.31	144.3	143.9	144.4	10.873 (10.45)	10.85 (10.88)	8.701 (10.88)	0.0
	0.10	688.57	689.5	689.6	640.7	36.021	36.01	35.62	36.09	5.341 (4.657)	5.382 (5.422)	4.338 (5.442)	0.0
	0.14	191.07	191.6	191.6	166.8	18.346	18.34	17.94	18.41	3.731 (2.884)	3.805 (3.857)	3.086 (3.887)	0.0
0.5	0.05	2,048.7	2051.0	2051.0	1989.0	40.657	40.67	40.50	40.84	6.243 (5.765)	6.163 (6.184)	4.948 (6.214)	0.0
	0.10	139.08	139.8	139.8	124.3	10.025	10.05	9.888	10.21	2.957 (2.893)	2.885 (3.044)	2.436 (3.107)	0.0
	0.14	39.790	40.21	40.23	32.36	5.036	5.068	4.903	5.209	1.999 (1.186)	2.080 (2.131)	1.705 (2.219)	0.0

*numbers in parenthesis denote the shear stress values obtained from the stress equilibrium equations.

†from Reference [22]

following orthotropic material properties, typical of aragonite crystals (converted from elastic constants given in [22] to engineering constants), are used.

$$\begin{aligned} E_1 &= 20.83 \times 10^6 \text{ psi}, E_2 = 10.94 \times 10^6 \text{ psi} \\ G_{12} &= 6.10 \times 10^6 \text{ psi}, G_{13} = 3.71 \times 10^6 \text{ psi}, G_{23} = 6.19 \times 10^6 \text{ psi} \\ \nu_{12} &= 0.44, \nu_{21} = 0.23. \end{aligned} \tag{29}$$

The elastic constant c_{11} used in Table 2 has the value of 23.2×10^6 psi. The following nondimensionalized deflections and stresses are tabulated in Table 1:

$$\begin{aligned} \bar{w} &= w\left(\frac{a}{2}, \frac{b}{2}, z\right)(h^3E_2/q_0a^4) \\ \bar{\sigma}_i &= \sigma_i\left(\frac{a}{2}, \frac{b}{2}, \pm\frac{h}{2}\right)(h^2/q_0a^2), i = 1, 2 \\ \bar{\sigma}_6 &= \sigma_6\left(0, 0, \pm\frac{h}{2}\right)(h^2/q_0a^2) \\ \bar{\sigma}_4 &= \sigma_4\left(\frac{a}{2}, 0, 0\right)(h/q_0a) \\ \bar{\sigma}_5 &= \sigma_5\left(0, \frac{b}{2}, 0\right)(h/q_0a). \end{aligned} \tag{30}$$

Two pairs of transverse shear stresses, one obtained from the constitutive equations and the other from equilibrium equations are presented in the tables. In the first-order shear deformation theory, the shear correction factors are assumed to be $L_1^2 = K_2^2 = 5/6$. The following conclusions can be drawn from the results of Tables 1 and 2:

(1) Even for the isotropic plates the effect of transverse shear deformation is significant. The classical plate theory (CPT) under predicts (for $a/b = 1$) the deflections by 4.9% at $a/h = 10$ and 17% at $a/h = 5$, and stress σ_1 by 0.7% at $a/h = 10$ and 2.6% at $a/h = 5$ when compared to the higher-order shear deformation plate theory (HSDPT); see Table 1.

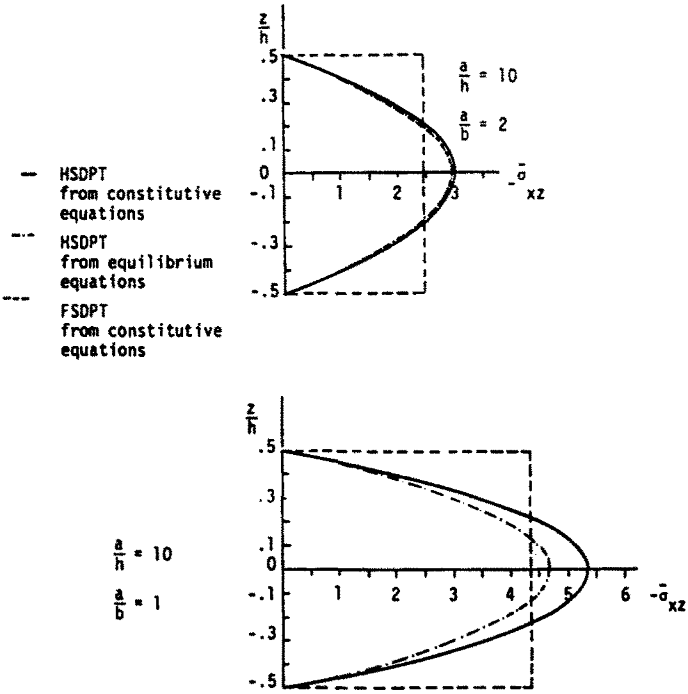

Fig. 1. Distribution of the transverse shear stress across the thickness of simply supported rectangular plates under uniformly distributed transverse load (orthotropic case).

(2) The first-order shear deformation plate theory (FSDPT) is quite accurate when the transverse deflections are concerned. But the stresses are no better than those predicted by CPT; see Tables 1 and 2.

(3) The transverse shear stresses predicted by the constitutive equations (8) of the higher-order theory are the most accurate of the three plate theories when compared to the exact solution of Srinivas and Rao[22]; see Table 2 and Fig. 1. It is interesting to note that FSDPT and CPT give more accurate transverse shear stresses than HSDPT when the stress equilibrium equations of 3-D elasticity theory are used:

$$\sigma_{xz} = -\int_{-h/2}^{h/2} (\sigma_{xx,x} + \sigma_{xy,y})\,dz$$

etc. (see Appendix 1).

(4) The infinite series for deflections converges faster than those for stresses; the convergence is slower for thick plates than for thin plates; see Table 1.

In summary, the higher-order theory yields more accurate distribution of stresses, especially shear stresses, when compared to the other plate theories. This feature of the higher-order theory is of considerable interest in the analysis of laminated composite plates, because an accurate prediction of the interlaminar shear stresses enables an accurate determination of the strength and failure of laminates.

Natural vibration

The numerical results of the natural vibration of isotropic ($\nu = 0.3$) and orthotropic (see eqn (29) for material properties) square plates are presented in Tables 3 and 4, resp. The results are compared with the exact solutions of the three-dimensional elasticity theory[22–24]. In Table 4 the first three eigenvalues obtained by the present theory are compared with the exact values, and the values obtained by FSDPT and CPT. From the results presented in Tables 3 and 4 the following observations can be made:

Table 3. Comparison of natural frequencies, $\bar{\omega} = \omega h(\sqrt{\rho/G})$, of isotropic ($\nu = 0.3$) plates ($a/h = 10$)

m	n	b/a = 1				m	n	b/a = √2			
		Exact [23]	HSDPT	FSDPT	CPT			Exact [24]	HSDPT	FSDPT	CPT
1	1	0.0932	0.0931	0.0930	0.0955 (0.0963)+	1	1	0.704	0.7038	0.7036	0.7180 (0.7224)
2	2	0.226	0.2222	0.2219	0.2360 (0.2408)	1	2	1.376	1.3738	1.3729	1.4273 (1.4448)
2	2	0.3421	0.3411	0.3406	0.3732 (0.3853)	2	1	2.018	2.0141	2.0123	2.1281 (2.1671)
1	3	0.4171	0.4158	0.4149	0.4629 (0.4816)	1	3	2.431	2.4263	2.4235	2.5908 (2.6487)
2	3	0.5239	0.5221	0.5205	0.5951 (0.6261)	2	2	2.634	2.6283	2.6250	2.8207 (2.8895)
1	4	-	0.6545	0.6520	0.7668 (0.8187)	2	3	3.612	3.6013	3.5948	3.9575 (4.0935)
3	3	0.6889	0.6862	0.6834	0.8090 (0.8669)	1	4	3.809	3.7891	3.7818	4.1822 (4.3343)
2	4	0.7511	0.7481	0.7446	0.8926 (0.9632)	3	1	3.987	3.9748	3.9666	4.4062 (4.5751)
3	4	-	0.8949	0.8896	2.0965 (2.2040)	3	2	4.535	4.5138	4.5089	5.0729 (5.2974)
1	5	0.9268	0.9230	0.9174	1.1365 (1.2521)	2	4	4890	4.8737	4.8608	5.5133 (5.7790)
2	5	-	2.0053	0.9984	1.2549 (1.3966)	3	3	5.411	5.3915	5.3754	6.1680 (6.5014)
4	4	1.0889	1.0847	1.0764	1.3716 (1.5411)	1	5	5.411	5.3915	5.3754	6.1680 (6.5014)
3	5	-	1.1361	1.1268	1.4475 (1.6374)	2	5	6.409	6.3846	6.3609	7.4563 (7.9462)

+numbers in parenthesis denote natural frequencies obtained by omitting the rotatory inertia.

Table 4. Comparison of natural frequencies, $\bar{\omega} = \omega h(\sqrt{\rho/c_{11}})$, of an orthotropic square plate ($a/h = 10$)

		Exact [22]			HSDPT			FSDPT			CPT
m	n	I	II*	III	I	II*	III	I	II*	III	I
1	1	0.0474	1.3077	1.6530	0.0474	1.3086	1.6550	0.0474	1.3159	1.6646	0.0493 (0.0497)+
1	2	0.1033	1.3331	1.7160	0.1033	1.3339	1.7209	0.1032	1.3410	1.7305	0.1095 (0.1120)
2	1	0.1188	1.4205	1.6805	0.1189	1.4216	1.6827	0.1187	1.4285	1.6921	0.1327 (0.1354)
2	2	0.1694	1.4316	1.7509	0.1695	1.4323	1.7562	0.1692	1.4393	1.7655	0.1924 (0.1987)
1	3	0.1888	1.3765	1.8115	0.1888	1.3772	1.8210	0.1884	1.3841	1.8305	0.2070 (0.2154)
3	1	0.2180	1.5777	1.7334	0.2184	1.5789	1.7361	0.2178	1.5857	1.7450	0.2671 (0.2779)
2	3	0.2475	1.4596	1.8523	0.2477	1.4603	1.8622	0.2469	1.4670	1.8714	0.2879 (0.3029)
3	2	0.2624	1.5651	1.8195	0.2629	1.5658	1.8255	0.2619	1.5725	1.8341	0.3248 (0.3418)
1	4	0.2969	1.4372	1.9306	0.2969	1.4379	1.9466	0.2959	1.4445	1.9560	0.3371 (0.3599)
4	1	0.3319	1.7179	1.8548	0.3330	1.7186	1.8588	0.3311	1.7265	1.8657	0.4471 (0.4773)
3	3	0.3320	1.5737	1.9289	0.3326	1.5744	1.9395	0.3310	1.5812	1.9479	0.4172 (0.4470)
2	4	0.3476	1.5068	1.9749	0.3479	1.5076	1.9912	0.3463	1.5141	2.0002	0.4152 (0.4480)
4	2	0.3070	1.6940	1.9447	0.3720	1.6947	1.9514	0.3696	1.7022	1.9586	0.5018 (0.5415)

* Pure thick-twist modes

+ Numbers in parenthesis indicate frequencies obtained by omitting the rotatory inertia.

(1) The classical plate theory overestimates the frequencies. The errors increase with increasing mode numbers.

(2) The frequencies predicted by FSDPT are fairly accurate; the error increases with increasing mode number.

(3) The frequencies predicted by HSDPT are the most accurate of all.

(4) The effect of transverse shear deformation increases with increasing mode number.

CONCLUSIONS

A refined nonlinear shear deformation theory of flat plates is presented. The theory accounts for (a) zero traction boundary conditions on the top and bottom faces of the plate, (b) cubic variation of in-plane displacements through thickness (hence, a parabolic distribution of transverse shear stresses through thickness), and (c) the von Karman strains. Additional features of the theory are that no shear correction factors are used in the theory, and the resulting equations of motion include the same variables as in the first-order shear deformation theory. Exact solutions for the case of infinitesimal displacements are presented for bending and free vibration of simply supported rectangular plates of isotropic as well as orthotropic materials. The solutions of the higher-order theory are found to be in excellent agreement with the exact solutions of the three-dimensional theory of elasticity. The numerical results should serve as references for those who wish to develop a finite-element model of the higher-order theory described herein. Extension of the present theory to laminated anisotropic plates is presented by the author (see[25]).

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APPENDIX

Transverse shear stresses from stress equilibrium equations

The transverse normal and shear stresses obtained from the stress equilibrium equations for the classical plate theory, the first-order shear deformation theory, and the higher-order theory (for simply supported, orthotropic rectangular plates) are given below:

(1) *Classical plate theory*

$$\sigma_z = -\frac{h^3}{48}\left\{\left[1+\left(\frac{2z}{h}\right)^3\right]+6\left[1+\left(\frac{2z}{h}\right)\right]\right\}[\alpha^4 Q_{11} + 2\alpha^2\nu^2(2G_{12}+Q_{12}) + \beta^4 Q_{22}]W \sin\alpha x \sin\beta y$$

$$\sigma_{xz} = \frac{h^2}{8}\left[1-\left(\frac{2z}{h}\right)^2\right][\alpha^3 Q_{11} + (2G_{12}+Q_{12})\alpha\beta^2]W \cos\alpha x \sin\beta y$$

$$\sigma_{yz} = \frac{h^2}{8}\left[1-\left(\frac{2z}{h}\right)^2\right][\alpha^2\beta(2G_{12}+Q_{12}) + \beta^3 Q_{22}]W \sin\alpha x \cos\beta y$$

where Q_{ij} are given by eqn (86), and W denotes the amplitude in eqn (22).

(2) *First-order shear deformation theory*

$$\sigma_z = \frac{h^3}{45}\left\{\left[1+\left(\frac{2z}{h}\right)^3\right]+6\left[1+\left(\frac{2z}{h}\right)\right]\right\}\{[\alpha^3 Q_{11} + \alpha\beta^2(2G_{12}+Q_{12})]X + [(2G_{12}+Q_{12})\alpha^2\beta + \beta^3 Q_{22}]Y\} \sin\alpha x \sin\beta y$$

$$\sigma_{xz} = -\frac{h^2}{8}\left[1-\left(\frac{2z}{h}\right)^2\right][(\alpha^2 Q_{11} + \beta^2 G_{12})X + (G_{12}+Q_{12})\alpha\beta Y] \cos\alpha x \sin\beta y$$

$$\sigma_{yz} = -\frac{h^2}{8}\left[1-\left(\frac{2z}{h}\right)^2\right][(\alpha^2 G_{12} + \beta^2 Q_{22})Y + (G_{12}+Q_{12})\alpha\beta X] \sin\alpha x \cos\beta y$$

(3) *Higher-order theory (present)*

$$\bar{\sigma}_z = \sigma_z - \frac{h^3}{480}\left\{\left[1+\left(\frac{2z}{h}\right)^5\right]+10\left[1+\left(\frac{2z}{h}\right)\right]\right\}\{[\alpha^4 Q_{11} + 2(2G_{12}+Q_{12})\alpha^2\beta^2 + \beta^4 Q_{22}]W$$
$$+ [\alpha^3 Q_{11} + \alpha\beta^2(2G_{12}+Q_{12})]X[\alpha^2\beta(2G_{12}+Q_{12}) + \beta^3 Q_{22}]Y\} \sin\alpha x \sin\beta y$$

$$\bar{\sigma}_x = \sigma_{xz} + \frac{h^2}{48}\left[1-\left(\frac{2z}{h}\right)^4\right]\{[\alpha^3 Q_{11} + (2G_{12}+Q_{12})\alpha\beta^2]W$$
$$+ (\alpha^2 Q_{11} + \beta^2 G_{12})X + \alpha\beta(G_{12}+Q_{12})Y] \cos\alpha x \sin\beta y$$

$$\bar{\sigma}_{yz} = \sigma_{yz} + \frac{h^2}{48}\left[1-\left(\frac{2z}{h}\right)^4\right]\{[(2G_{12}+Q_{12})\alpha^2\beta + \beta^3 Q_{22}]W$$
$$+ \alpha\beta(G_{12}+Q_{12})X + (\alpha^2 G_{12} + \beta^2 Q_{22})Y\} \sin\alpha x \cos\beta y$$

where σ_z, σ_{xz} and σ_{yz} are the expressions given for the first-order shear deformation theory.